LIE GROUPS AND LIE ALGEBRAS MIDTERM EXAMINATION

Total marks: 100

The base field F, in all questions, has characteristic zero and is algebraically closed.

(1) State whether each of the following statements is true or false, and justify your answer, by quoting a result taught in class or by constructing an example/counterexample.

(a) A solvable Lie algebra must have nonzero center.

(b) Any two dimensional Lie algebra is solvable.

(c) Any three dimensional Lie algebra is either simple or solvable.

(d) There does not exist a four dimensional semisimple Lie algebra.

(e) The Killing form on a solvable Lie algebra is identically zero.

(8 x 5 = 40 marks)

(2) Describe (without proof) the irreducible representations $V(m)$ of $\mathfrak{sl}(2, F)$ (for every non negative integer m). Let $M = \mathfrak{sl}(3, F)$ and $L = \mathfrak{sl}(2, F)$. Then M contains a copy of L in its upper left hand 2×2 position. View M as an L-module via the adjoint representation, and prove that M, is isomorphic to $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$. (10 + 10 = 20 marks)

(3) Let L be a Lie algebra such that $Z(L) = Rad(L)$ (such a Lie algebra is called reductive). Show that L is completely reducible as an $ad(L)$-module. Also prove that, L is isomorphic to the direct sum $Z(L) \oplus [L, L]$, with $[L, L]$ semisimple. (10 + 10 = 20 marks)

(4) Describe a Cartan decomposition for the semisimple Lie algebra $\mathfrak{sl}(n, F)$. Write down the set of roots for the decomposition. For any two non proportional roots α, β, calculate explicitly the α string through β, and compute the Cartan integers $\beta(h_\alpha)$. (10 + 10 = 20 marks)

Date: February 15, 2017.